

On the exponent of the automorphism group of a compact Riemann surface

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To Ernst-Ulrich Gekeler in gratitude for his support and for everything that I learned from him.

Abstract

Let X be a compact Riemann surface of genus $g \geq 2$, and let $Aut(X)$ be its group of automorphisms. We show that the exponent of $Aut(X)$ is bounded by $42(g-1)$. We also determine explicitly the infinitely many values of g for which this bound is reached and the corresponding groups. Finally we discuss related questions for subgroups G of $Aut(X)$ that are subject to additional conditions, for example being solvable.

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1. Introduction

Throughout the paper X will be a compact Riemann surface of genus $g \geq 2$. We write $Aut(X)$ for the full group of conformal automorphisms of X .

The order of a group G is denoted by $|G|$, the cyclic group of order n by C_n , and the neutral element in a group by ι .

Theorem 1.1. (Hurwitz) *Let X be a compact Riemann surface of genus $g \geq 2$. Then*

$$|Aut(X)| \leq 84(g-1).$$

Moreover, it is known that there are infinitely many values of g for which the bound in Theorem 1.1 is reached. For proofs of all this see for example [A, pp.46] or [Br, Theorem 3.17].

A group $Aut(X)$ that reaches the bound $|Aut(X)| = 84(g-1)$ is called a **Hurwitz group**. In Section 3 we will provide some more details on these groups.

After this one may of course ask how big a group G that satisfies some additional properties can be if it acts as a group of automorphisms on a Riemann surface of genus g (i.e. if $G \subseteq Aut(X)$). From the vast literature we select some results that

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are relevant for this paper.

Theorem 1.2. *Let X be a compact Riemann surface of genus $g \geq 2$, and let G be a subgroup of $\text{Aut}(X)$.*

- (a) *If G is solvable, then $|G| \leq 48(g - 1)$. There are infinitely many values of g for which this bound is reached.*
- (b) *If G is supersolvable, then for $g \geq 3$ we have $|G| \leq 18(g - 1)$. There are infinitely many values of g for which this bound is reached. The biggest supersolvable group of automorphisms for genus 2 has order 24.*
- (c) *If G is nilpotent, then $|G| \leq 16(g - 1)$. There are infinitely many values of g for which this bound is reached.*
- (d) *If G is abelian, then $|G| \leq 4g + 4$. For each $g \geq 2$ there are abelian groups of order $4g + 4$ acting as automorphisms on a Riemann surface of genus g .*

Proof. (a) The bound results from the facts that Hurwitz groups are not solvable (see Corollary 3.2 (a) below) and that $48(g - 1)$ is the next possible size of $\text{Aut}(X)$ [Br, Lemma 3.18]. Groups that reach the bound were constructed in [Ch] and [G1]. See also [G2] for improvements and minor corrections.

(b) The papers [Z2] and [GM1] seem to have been written independently and at almost the same time. In [Z2] the condition for g to reach the bound contains an error, which is pointed out in [GM1] and also corrected in [Z3].

(c) [Z1, Theorems 1.8.4 and 2.1.2]

(d) See [G1, p.271]. The paper [M1] contains more precise information, namely on page 711 for each abelian group the minimal genus for which it can occur in $\text{Aut}(X)$. \square

Given $G \subseteq \text{Aut}(X)$, it is in general very difficult to decide whether G equals $\text{Aut}(X)$ or is a proper subgroup. In the special case $|G| = 48(g - 1)$ we automatically have equality, as by [Br, Lemma 3.18] the only bigger order is $84(g - 1)$, which is not a multiple of $48(g - 1)$.

A statement as in Theorem 1.2 for cyclic subgroups of $\text{Aut}(X)$ is of course equivalent to a statement about element orders. In [H, Theorem 6] for each n the minimum genus for an automorphism of order n is given. From this one can get the following classical result. Alternatively, see [G1, p.270].

Theorem 1.3. (Wiman) *Let X be a compact Riemann surface of genus $g \geq 2$. Then the element orders of $\text{Aut}(X)$ are bounded by $4g + 2$.*

For each $g \geq 2$ there exists an X of genus g such that $\text{Aut}(X)$ contains elements of order $4g + 2$.

The paper [N] classifies all Riemann surfaces with an automorphism of order $\geq 3g$.

In this paper we investigate a problem that does not seem to have been treated yet in the literature, namely bounding the exponent of $\text{Aut}(X)$ (or of a subgroup $G \subseteq \text{Aut}(X)$) in terms of the genus.

2. The exponent of a group

The **exponent** $\exp(G)$ of a finite group G is the least common multiple of all element orders. Equivalently, $\exp(G)$ is the smallest positive integer e such that $\sigma^e = \iota$ for all $\sigma \in G$.

We leave the following three facts as easy exercises.

Lemma 2.1.

$$\exp(G) = \prod_p \exp(G_p),$$

where the product is over all primes p dividing $|G|$ and G_p is a Sylow p -subgroup of G .

Lemma 2.2. *The exponent of a finite p -group P is the biggest element order. In particular, $\exp(P) = |P|$ if and only if P is cyclic.*

Corollary 2.3.

- (a) $\exp(G) = |G|$ if and only if all Sylow subgroups of G are cyclic.
- (b) $\exp(G) = \frac{1}{2}|G|$ if and only if for all odd primes p the Sylow p -subgroups of G are cyclic and the (non-cyclic) Sylow 2-subgroup has a cyclic subgroup of index 2.

Finite groups whose Sylow subgroups are all cyclic are called **Z-groups**, possibly from the German word zyklisch or perhaps in honor of H. Zassenhaus, who completely described the structure of these groups. For many purposes the form in [R, Theorem 10.1.10] is better than the one in [Za, Satz 5].

Theorem 2.4. (Zassenhaus) *A Z-group that is not cyclic can be written as a semidirect product*

$$C_m \rtimes C_n$$

where $(m, n) = 1$ and m is odd. In particular, such a group is metacyclic and hence supersolvable.

The most important type of groups for our paper are the ones from Corollary 2.3 (b), and the most important instance of such groups is the following.

Example 2.5. Let p be an odd prime. Then

$$|PSL_2(\mathbb{F}_p)| = \frac{p^3 - p}{2} \quad \text{and} \quad \exp(PSL_2(\mathbb{F}_p)) = \frac{p^3 - p}{4}.$$

Actually, finite groups G with $\exp(G) = \frac{1}{2}|G|$ have been completely classified, the solvable ones in [Za, Satz 7] and the nonsolvable ones in [S] and [W2]. See also the first page of [W1] where it is explained why the different types discussed in [S] and [W2] cover all possible cases. The final summary is

Theorem 2.6. (Suzuki, Wong) [W2, Theorem 2] *Let G be a nonsolvable finite group in which all Sylow subgroups of odd order are cyclic and a Sylow 2-subgroup has a cyclic subgroup of index 2. Then G has a normal subgroup G_1 such that $[G : G_1] \leq 2$ and*

$$G_1 = L \times M,$$

where L is isomorphic to $SL_2(\mathbb{F}_p)$ or $PSL_2(\mathbb{F}_p)$ for some prime $p \geq 5$, and M is a Z -group whose order is prime to that of L .

3. Hurwitz groups

In this section we collect the necessary details about Hurwitz groups. See [C] for more background.

The first result comes from the fact that Hurwitz groups are exactly the non-trivial finite quotients of the triangle group $\Gamma(0; 2, 3, 7)$. See for example [Br, Theorem 3.17]. Actually, Section 3 of [Br] is a compact survey (with references) that covers everything we need about triangle groups.

Theorem 3.1. *A non-trivial, finite group is a Hurwitz group if and only if it can be generated by two elements σ and τ subject to*

$$\sigma^2 = \tau^3 = (\sigma\tau)^7 = \iota$$

and some other relations.

Corollary 3.2. *Let G be a Hurwitz group. Then*

- (a) *G has no non-trivial abelian quotient group. So the commutator group G' equals G , and G is not solvable.*
- (b) *Every non-trivial quotient group of G has order divisible by 42.*
- (c) *Every non-trivial quotient of G is again a Hurwitz group.*

Proof. We write $\tilde{\sigma}$ and $\tilde{\tau}$ for the images of σ and τ in the quotient.

(a) If $\tilde{\sigma}$ and $\tilde{\tau}$ commute, then $(\tilde{\sigma}\tilde{\tau})^6 = \tilde{\iota}$, and consequently $\tilde{\sigma}\tilde{\tau} = \tilde{\iota}$, $\tilde{\sigma} = \tilde{\iota}$, and $\tilde{\tau} = \tilde{\iota}$.

(b) Similarly, if 42 does not divide the order of the quotient group, then one, and hence all, of $\tilde{\sigma}$, $\tilde{\tau}$, $\tilde{\sigma}\tilde{\tau}$ must equal $\tilde{\iota}$.

(c) This is immediate from Theorem 3.1. \square

The question which finite groups are Hurwitz groups is far from being completely solved. Even for finite simple groups the answer is quite irregular (see [C]). We are interested in a special type of group.

Theorem 3.3. (Macbeath) [Mb, Theorem 8] *The group $PSL_2(\mathbb{F}_q)$ is a Hurwitz group if and only if*

(i) $q = 7$,

(ii) $q = p$, a prime, with $p \equiv \pm 1 \pmod{7}$,

(iii) $q = p^3$, where p is a prime with $p \equiv \pm 2$ or $\pm 3 \pmod{7}$,

and for no other values of q .

In cases (i) and (iii) there is only one Riemann surface on which G acts as a Hurwitz group. In case (ii) there are three Riemann surfaces for each G .

In contrast, we point out the following easy result.

Theorem 3.4. $SL_2(\mathbb{F}_p)$ is not a Hurwitz group.

Proof. Obviously the involution σ (in Theorem 3.1) of a Hurwitz group cannot be central. But $SL_2(\mathbb{F}_p)$ has exactly one involution, which thus is of course central. Actually, it is the negative of the unit matrix. \square

By Corollary 3.2 the exponent of a Hurwitz group has to be divisible by 42. For use in later sections we refine this statement. To that end we need the following group theoretic result.

Theorem 3.5. *Let G be a non-abelian, simple group of order $2^a 3^b 5^c 7^d$ with abelian Sylow 2-subgroup. Then G must be among the groups $PSL_2(\mathbb{F}_{p^n})$ with $p \in \{2, 3, 5, 7\}$.*

Proof. By [Wa, Theorem I] and the remarks immediately after it, a non-abelian finite simple group G that has abelian Sylow 2-subgroups and is not of type $PSL_2(\mathbb{F}_{p^n})$ must either be the Janko group J_1 of order $175,560 = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$ or it must contain a subgroup $PSL_2(\mathbb{F}_{3^{2n+1}})$ with $n > 0$.

By an elementary number theoretic argument we show now that the order of such a group is always divisible by a prime $p > 7$. Obviously, $(3^{2n+1})^2 - 1$ is congruent

to 8 (mod 16), -1 (mod 3), and 3 (mod 5). So if it is not divisible by any prime $p > 7$, it must be of the form

$$(3^{2n+1})^2 - 1 = 8 \cdot 7^m \quad \text{with } m > 1.$$

Calculating modulo 9 shows that m must be divisible by 3, say $m = 3k$. And modulo 7 we see that necessarily $n = 3h + 1$. Thus $(3^{4h+2}, 2 \cdot 7^k)$ is an integral solution of $x^3 - y^3 = 1$. But writing this as $(x - y)(x^2 + xy + y^2) = 1$ clearly shows the impossibility of such integral solutions. \square

Doubtlessly, the groups in Theorem 3.5 can be completely determined, and most likely this is known. But the crude version above suffices for our goal, namely to prove

Theorem 3.6. *There are no Hurwitz groups of exponent $2 \cdot 3 \cdot 7^n$. In particular, there are no Hurwitz groups of exponent 42.*

Proof. Assume that G is such a group. Let N be a maximal normal subgroup. By Corollary 3.2 (b) and (c), G/N is a simple Hurwitz group, whose exponent is of the same form, possibly with a smaller n .

The Sylow 2-subgroup of G/N has exponent 2; so in particular it must be abelian. Hence $G/N \cong PSL_2(\mathbb{F}_{p^n})$ with $p \in \{2, 3, 5, 7\}$ by Theorem 3.5. Now Theorem 3.3 leaves only the following four candidates for G/N , which however all fail: $PSL_2(\mathbb{F}_7)$ has exponent $2^2 \cdot 3 \cdot 7$; $PSL_2(\mathbb{F}_8)$ has exponent $2 \cdot 3^2 \cdot 7$; $PSL_2(\mathbb{F}_{27})$ has exponent $2 \cdot 3 \cdot 7 \cdot 13$; and $PSL_2(\mathbb{F}_{125})$ has exponent $2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$. \square

4. The main result

Theorem 4.1. (Main Theorem, first version) *Let X be a compact Riemann surface of genus $g \geq 2$. Then*

$$\exp(\text{Aut}(X)) \leq 42(g - 1).$$

Equality holds if and only if $G = \text{Aut}(X)$ is a Hurwitz group with $\exp(G) = \frac{1}{2}|G|$.

Proof. If $\exp(\text{Aut}(X)) = |\text{Aut}(X)|$, then $\text{Aut}(X)$ is supersolvable by Theorem 2.4. By Theorem 1.2 (b) therefore $\exp(\text{Aut}(X))$ is significantly smaller than $42(g - 1)$.

In all other cases we have $\exp(\text{Aut}(X)) \leq \frac{1}{2}|\text{Aut}(X)| \leq \frac{84}{2}(g - 1)$, with equality if and only if $\exp(\text{Aut}(X)) = \frac{1}{2}|\text{Aut}(X)|$ and $|\text{Aut}(X)| = 84(g - 1)$. \square

Remark 4.2. Theorem 4.1 already shows that there are infinitely many values of g for which the bound $42(g - 1)$ cannot be reached, simply because there are no Hurwitz groups for these g . Take for example $g = 7^n + 1$. A group of order $84 \cdot 7^n$ has

a normal Sylow 7-subgroup, so it is solvable and thus cannot be a Hurwitz group. See [A, Chapter 5] for more sequences of g without Hurwitz groups.

On the other hand, there are infinitely many values of g for which at least one surface reaches $42(g-1)$. For example with the groups in Theorem 3.3 (i) and (ii). In the remainder of this section we want to show that these examples are the only ones.

Theorem 4.3. *The only Hurwitz groups with $\exp(G) = \frac{1}{2}|G|$ are the groups $PSL_2(\mathbb{F}_p)$ where $p = 7$ or p is a prime that is congruent to ± 1 modulo 7.*

Proof. As a complete classification of all Hurwitz groups is not known, and would almost certainly be very complicated anyway, we start with the other condition. Let G be a non-solvable group with $\exp(G) = \frac{1}{2}|G|$. These are completely classified in Theorem 2.6.

If G moreover is a Hurwitz group, we must have $G = G_1$ in that theorem by Corollary 3.2. Furthermore, the Z -group M in that theorem must be trivial, for otherwise we could map from G to M and from there to an abelian quotient, contradicting Corollary 3.2. So we are left with the possibilities $G \cong PSL_2(\mathbb{F}_p)$ or $G \cong SL_2(\mathbb{F}_p)$. But the second possibility is excluded by Theorem 3.4. Finally we apply Theorem 3.3. \square

Combining Theorem 4.1 and Theorem 4.3 we obtain

Theorem 4.4. (Main Theorem, final version) *Let X be a compact Riemann surface of genus $g \geq 2$. Then*

$$\exp(\text{Aut}(X)) \leq 42(g-1).$$

This bound can be reached if and only if

$$g = \frac{p^3 - p}{168} + 1$$

where $p = 7$ or p is a prime that is congruent to ± 1 modulo 7. The only Riemann surface of genus 3 that reaches the bound is the Klein quartic, whose automorphism group is isomorphic to $PSL_2(\mathbb{F}_7)$. If $p \equiv \pm 1 \pmod{7}$, there are 3 non-isomorphic Riemann surfaces X of genus $g = \frac{p^3 - p}{168} + 1$ with $\exp(\text{Aut}(X)) = 42(g-1)$. In every case

$$\text{Aut}(X) \cong PSL_2(\mathbb{F}_p).$$

5. Solvable groups

In accordance with Theorem 1.2 we now try to find upper bounds on $\exp(G)$ for the $G \subseteq \text{Aut}(X)$ that are subject to additional conditions.

The following partial result, which might be interesting in its own right, will be used repeatedly.

Proposition 5.1. *Let X be a compact Riemann surface of genus $g \geq 2$, and let $G \subseteq \text{Aut}(X)$ be a Z -group. Then*

$$|G| < 16(g - 1).$$

Proof. For $g = 2$ there are four groups of order ≥ 16 [Br, p.77], but none of them has a cyclic Sylow 2-subgroup.

So we can suppose from now on that $g \geq 3$. Since Z -groups are of course metabelian, by [ChP] we have $|G| \leq 16(g - 1)$ with two possible exceptions, namely $|G| = 48$ for $g = 3$ and $|G| = 80$ for $g = 5$. But by Theorem 1.2 (b) these two metabelian exceptions cannot be supersolvable, and hence in particular not Z -groups.

By [ChP] all metabelian groups G of order $16(g - 1)$ are quotients of $\Gamma(0; 2, 4, 8)$. If such a G is a Z -group, then Theorem 2.4 implies that G , and hence also $\Gamma(0; 2, 4, 8)$ has a cyclic quotient of order 16, which is clearly impossible. \square

Remark 5.2. We don't know what could be a sharp bound in Proposition 5.1. In any case there are infinitely many Z -groups of order $10(g - 1)$. Namely, by [BJ, Theorem 1] for every big enough prime p with $p \equiv 1 \pmod{5}$ there exists a Riemann surface X of genus $p + 1$ such that $\text{Aut}(X)$ contains $G \cong C_p \rtimes C_{10}$.

Proposition 5.3. *Let X be a compact Riemann surface of genus $g \geq 3$, and let G be a solvable subgroup of $\text{Aut}(X)$. Then*

$$\exp(G) \leq 16(g - 1).$$

Proof. If $\exp(G) < \frac{1}{2}|G|$, Theorem 1.2 (a) implies $\exp(G) \leq \frac{1}{3}|G| \leq 16(g - 1)$. And if $\exp(G) = |G|$, we even have $\exp(G) < 16(g - 1)$ by Proposition 5.1.

Solvable groups G with $\exp(G) = \frac{1}{2}|G|$ have been completely classified in [Za, Satz 7]. We only need the following key fact from the proof: If $\exp(G) = \frac{1}{2}|G|$, then G contains a normal Z -group G_1 such that G/G_1 is isomorphic to C_2 or A_4 or S_4 .

If $[G : G_1] = 2$, we have $\exp(G) = |G_1| < 16(g - 1)$ by Proposition 5.1. In the remaining cases we have to show that $|G| \leq 32(g - 1)$. By [Br, Lemma 3.18] there are only three possible orders of solvable groups bigger than $32(g - 1)$, namely $48(g - 1)$, $40(g - 1)$ and $36(g - 1)$. Correspondingly, we have to show that G cannot be a quotient of $\Gamma(0; 2, 3, 8)$, $\Gamma(0; 2, 4, 5)$ or $\Gamma(0; 2, 3, 9)$.

Obviously, the only finite quotient of $\Gamma(0; 2, 4, 5)$ of order prime to 5 is C_2 . So $\Gamma(0; 2, 4, 5)$ cannot have a quotient G that has a quotient A_4 or S_4 . Likewise, $\Gamma(0; 2, 3, 8)$ has no quotient C_3 and hence no quotient G with $G/G_1 \cong A_4$, whereas $\Gamma(0; 2, 3, 9)$ has no quotient C_2 and hence no quotient G with $G/G_1 \cong S_4$.

For the remaining two cases we use that since G_1 is supersolvable, the elements in G_1 of odd order form a characteristic subgroup U of G_1 [R, Theorem 5.4.9]. The normality of G_1 in G implies that U is normal in G . Furthermore, if U is non-trivial and p_1 is the smallest prime divisor of $|U|$, by Zappa's Theorem [R, Theorem 5.4.8] U has a normal subgroup M of index p_1 . Since the Sylow p_1 -subgroups of U are cyclic, M is even characteristic in U , and hence normal in G .

If $G/G_1 \cong S_4$ we obtain a chain of normal subgroups

$$G/M \triangleright N \triangleright V \triangleright G_1/M \triangleright U/M \triangleright I$$

with factors $C_2, C_3, C_2 \times C_2, C_{2^e}, C_{p_1}$. Now assume moreover that G is a quotient of $\Gamma = \Gamma(0; 2, 3, 8)$. By [Br, Example 3.8] we have $\Gamma' \cong \Gamma(0; 3, 3, 4)$ and $\Gamma'' \cong \Gamma(0; 4, 4, 4)$ with $\Gamma/\Gamma' \cong C_2$ and $\Gamma'/\Gamma'' \cong C_3$. This implies that N is a quotient of Γ' and $N' = V$ and V is a quotient of Γ'' . In particular, V cannot have a quotient C_{p_1} .

On the other hand, U/M is a normal subgroup of N . Let C be its centralizer in N . As N/C can be embedded into the automorphism group of U/M , which is cyclic, we see $C \supseteq N' = V$. So U/M is central in V . This means that V is a direct product of its Sylow 2-subgroup and U/M . In particular, V has a quotient C_{p_1} .

The resolution of this contradiction is that U must be trivial. Consequently $48(g-1) = |G| = 3 \cdot 2^{e+3}$. Since $\exp(G) = 24(g-1)$, in that case G must contain an element of order $8(g-1)$. By Theorem 1.3 this is only possible if $8(g-1) \leq 4g+2$, i.e., if $g \leq 2$.

Similarly, if G is a quotient of $\Gamma = \Gamma(0; 2, 3, 9)$ with $G/G_1 \cong A_4$, we obtain that G' lies between G and G_1 with $G/G' \cong C_3$ and G' is a quotient of the commutator group $\Gamma' \cong \Gamma(0; 2, 2, 2, 3)$. Moreover, since 36 divides $|G|$, we have $U/M \cong C_3$, and as above this group is central in G'/M , leading to the contradiction that Γ' should have a quotient C_3 . \square

We don't know whether for $g > 2$ the bound in Proposition 5.3 can be reached, and if yes whether infinitely often.

For genus 2 we mention that the Bolza surface $y^2 = x^5 - x$ has automorphism group $GL_2(\mathbb{F}_3)$ of order 48 and exponent 24.

Remark 5.4. By [Z2, Theorem 4.1] or [GML, Lemma 4.1] the supersolvable group of order 24 for $g = 2$ has exponent 12.

If $g \geq 3$ and G is supersolvable but not a Z -group, then from Theorem 1.2 (b) we get $\exp(G) \leq \frac{1}{2}|G| \leq 9(g-1)$, which is smaller than the examples mentioned in Remark 5.2. This shows that bounding $\exp(G)$ for supersolvable groups G amounts to the same as bounding $|G|$ for Z -groups G .

A finite nilpotent group G is the direct product of its Sylow subgroups. Its exponent therefore is the biggest element order (see Lemmas 2.1 and 2.2). So by Theorem 1.3 we have $\exp(G) \leq 4g+2$. By [N, Theorem 1] there is a unique surface of genus g that has an automorphism of order $4g+2$. Putting all together we obtain

Theorem 5.5. *Let X be a compact Riemann surface of genus $g \geq 2$, and let G be a nilpotent subgroup of $\text{Aut}(X)$. Then*

$$\exp(G) \leq 4g + 2.$$

For every $g \geq 2$ there exists, up to isomorphism, exactly one Riemann surface of genus g that realizes this bound, namely

$$y^2 = x^{2g+1} - 1,$$

which has $\text{Aut}(X) \cong C_{4g+2}$.

Obviously the same result holds for abelian subgroups G of $\text{Aut}(X)$.

6. On $|G|/\exp(G)$

Finally, we investigate the case when $\exp(G)$ is as small as possible compared to $|G|$.

Theorem 6.1. *Let X be a compact Riemann surface of genus $g \geq 2$, and let G be a subgroup of $\text{Aut}(X)$. Then*

$$\frac{|G|}{\exp(G)} \text{ divides } 2(g-1).$$

Proof. This is a well-known consequence of the Hurwitz formula

$$2g - 2 = |G|(2h - 2) + \sum_{i=1}^r \frac{|G|}{|S_i|}(|S_i| - 1)$$

for the covering $X \rightarrow X/G$. Here P_1, \dots, P_r are the branch points on the genus h Riemann surface X/G , and S_i is the stabilizer of a point on X above P_i . Since S_i is always cyclic, $\frac{|G|}{\exp(G)}$ divides $\frac{|G|}{|S_i|}$. \square

Proposition 6.2. *Let X be a compact Riemann surface of genus $g \geq 2$, and let G be a subgroup of $\text{Aut}(X)$. If $|G|/\exp(G) = 2(g-1)$, then G must be solvable and $\exp(G) \leq 24$.*

Proof. If $|G|/\exp(G) = 2(g-1)$, then Theorem 1.1 implies $\exp(G) \leq 42$, with equality if and only if G is a Hurwitz group of exponent 42. But by Theorem 3.6 such groups do not exist.

The second biggest possible size of G is $|G| = 48(g-1)$ [Br, Lemma 3.18]. This shows $\exp(G) \leq 24$. Consequently, $\exp(G)$ cannot have more than 2 different prime

divisors. By Burnside's $p^m q^n$ -Theorem (see for example [R, Theorem 8.5.3]) this implies that G is solvable. \square

If $|G|/\exp(G) = 2(g-1)$, then $\exp(G)$ must of course be even and divisible by all primes that divide $g-1$.

Moreover, the cases $\exp(G) = 22, 16$ or 14 cannot occur in Proposition 6.2, because there are no G of order $44(g-1)$, $32(g-1)$ or $28(g-1)$ ([Br, Lemma 3.18]). On the other hand, the Bolza surface from Remark 5.2 shows that $\exp(G) = 24$ can occur, at least for $g = 2$. We don't know whether it can occur for $g > 2$. But we have the following general finiteness result.

Theorem 6.3. *There are only finitely many groups G that reach the bound $|G|/\exp(G) = 2(g-1)$ in Theorem 6.1.*

Proof. Let $|G|/\exp(G) = 2(g-1)$. Dividing the Hurwitz formula in the proof of Theorem 6.1 by $|G|/\exp(G)$, we obtain

$$1 = \exp(G)(2h-2) + \sum_{i=1}^r \frac{\exp(G)}{|S_i|}(|S_i| - 1).$$

Fix one of the remaining exponents 24, 20, 18, 12, 10, 8, 6, 4, 2. Then, since $|S_i|$ divides $\exp(G)$, there are only finitely many values h and r for which this equation has a solution. More precisely, we must have $h = 0$ and $r \leq 5$, as $h = 1$ would imply $r = 1$, $|S_1| = 2$, and $\exp(G) = 2$, which is not possible. By the theory of Fuchsian groups, G is a quotient of a group that is generated by $2h + r - 1$ elements. But by the affirmative solution to the restricted Burnside problem [Ze] there are only finitely many finite groups with a given number of generators and a given exponent. \square

Finally, the case $\exp(G) = 2$ can be completely settled. All groups of exponent 2 are abelian; so we do this in slightly more generality.

Theorem 6.4. *There are only five abelian groups G that reach the bound $|G|/\exp(G) = 2(g-1)$ in Theorem 6.1, namely*

- $C_2 \times C_2$ and $C_6 \times C_2$ for $g = 2$;
- $C_2 \times C_2 \times C_2$ and $C_4 \times C_4$ for $g = 3$;
- $C_2 \times C_2 \times C_2 \times C_2$ for $g = 5$.

Proof. If $G \subseteq \text{Aut}(X)$ is abelian, then $|G| \leq 4g+4$ by Theorem 1.2 (d). If moreover $|G|/\exp(G) = 2(g-1)$, this leaves only the possibilities $\exp(G) \in \{2, 4, 6\}$ for $g = 2$, $\exp(G) \in \{2, 4\}$ for $g = 3$, and $\exp(G) = 2$ for $g > 3$. So besides the groups listed in the theorem, the possible candidates are $C_4 \times C_2$ for $g = 2$, $C_4 \times C_2 \times C_2$ for $g = 3$,

and $(C_2)^r$, $r \geq 5$ for $g = 2^{r-2} + 1$. But [Ml, Theorem 4] shows that the minimum genus for the latter three types is bigger. \square

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